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A Cellular Triangle Containing a Specified Point

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Abstract. Let P be a set of finite points in the plane in general position, and let x be a point which is not contained in any of the lines passing through at least two points of P. A line l is said to be a *k-bisector* if both of the two closed half-planes determined by l contain at least k points of P. We show that if any line passing through x is a $\left\lceil \frac{|P|}{3} \right\rceil$-bisector and does not contain two or more points of P, then there exist three points p_1, p_2, p_3 of P such that $\Delta p_1 p_2 p_3$ contains x and does not contain points of P in its interior, and such that each of the lines passing through two of them is a $\left\lceil \frac{|P|}{3} \right\rceil$-bisector.

1. Introduction

For a subset V of the Euclidian plane $\mathbb{R}^2$, let $\mathrm{conv}(V)$ denote the convex hull of V. Let P be a finite set of points in $\mathbb{R}^2$. For a point $x \in \mathbb{R}^2$, the *depth* of x with respect to P, denoted by $\mathrm{dep}(P, x)$, is the minimal number of points of P conained in a closed half-plane determined by a line passing through x $\left(\text{clealy } \mathrm{dep}(P, x) \leq \frac{|P|}{2}\right)$. For three non-colinear points $x, y, z \in \mathbb{R}^2$, denote by $H^+(x; y, z)$ (resp. $H^-(x; y, z)$) the open half-plane which is determined by the line passing through y, z, and contains (resp. misses) x. Further let

$$P(x; y, z) := P \cap H^+(y; z, x) \cap H^+(z; x, y)$$

(see Fig. 1). A generalized version of Perles' rooted tree embedding problem (see [1]) is partially solved in [2] by making use of a property of finite point sets in the plane. Further in [2], the auther made the following conjecture, which seems contribute to the complete solution of the problem (the statement is slightly modified):

Conjecture 1. *Suppose that* $n = |P| \geq 3$, *and let* k_1, k_2, k_3 *be integers such that* $0 \leq k_1 \leq k_2 \leq k_3 \leq \frac{n-2}{2}$ *and* $k_1 + k_2 + k_3 = n - 3$. *Let* x *be a point of* $\mathbb{R}^2 - P$, *and suppose that* $P \cup \{x\}$ *is in general position and* $\mathrm{dep}(P, x) \geq k_3 + 1$. *Then there exist three non-collinear points* p_1, p_2, p_3 *of* P *such that* $P \cap \mathrm{conv}(\{p_1, p_2, p_3\}) =$

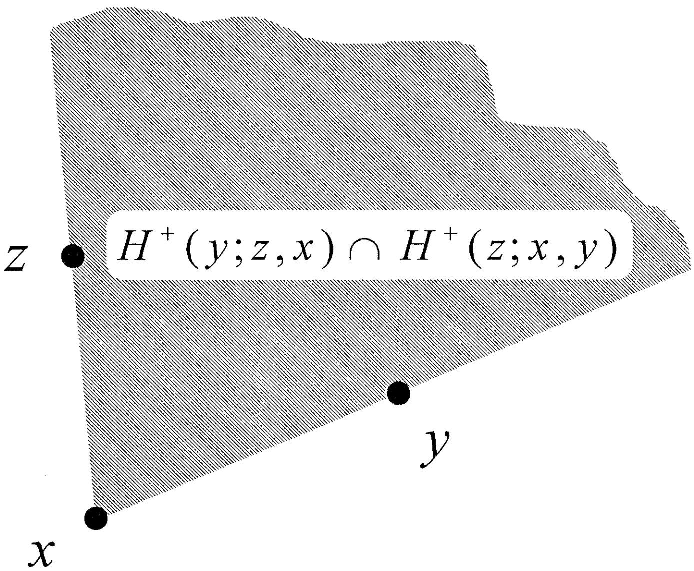

Fig. 1

$\{p_1, p_2, p_3\}$ *and the following inequalities hold*:

$$|P(p_1; p_2, p_3)| \leq k_1 \leq |P \cap H^-(p_1; p_2, p_3)|,$$
$$|P(p_2; p_3, p_1)| \leq k_2 \leq |P \cap H^-(p_2; p_3, p_1)|,$$
$$|P(p_3; p_1, p_2)| \leq k_3 \leq |P \cap H^-(p_3; p_1, p_2)|,$$

As a partial solution of the above conjecture, we show the following.

Theorem 1. *Let* P, k_1, k_2, k_3, *and* x *be as in Conjecture 1. Then there exist three non-collinear points* p_1, p_2, p_3 *of* P *such that* $x \in \mathrm{conv}(\{p_1, p_2, p_3\})$, $P \cap \mathrm{conv}(\{p_1, p_2, p_3\}) = \{p_1, p_2, p_3\}$, *and the following three inequalities hold*:

$$|P \cap H^-(p_1; p_2, p_3)| \geq k_1, \tag{1}$$
$$|P \cap H^-(p_2; p_3, p_1)| \geq k_2, \tag{2}$$
$$|P \cap H^-(p_3; p_1, p_2)| \geq k_3. \tag{3}$$

If a subset S of $\mathbb{R}^2$ contains no points of P in its interior, S is said to be *vacuum*. Further, following Kupitz [3], we call a convex polygon D *cellular* if D is vacuum and all vertices of D are points of P. Kupitz also introduced the notation of a *k-bisector*, which is a line l such that both of the two closed halfplanes determined by l contain at least k points of P. With this terminology, we can state a special case of Theorem 1 as follows:

Corollary 1. *Let* P *be a set of* n (≥ 3) *points in general position in the plane, and let* $x \in \mathbb{R}^2 - P$. *Suppose that each line passing through* x *is a* $\left\lceil \frac{n}{3} \right\rceil$*-bisector and does not* *contain two or more points of P. Then there exists a cellular triangle Δ of P containing x such that each edge of Δ is on a $\left\lceil \frac{n}{3} \right\rceil$-bisector.*

2. Proof

We first settle a special case of Theorem 1:

Claim 1. *Let P, k_1, k_2, k_3, x be as in Theorem 1. Suppose that there exist $q_1, q_2, q_3 \in P$ such that $\Delta q_1 q_2 q_3$ contains x, $\Delta x q_1 q_2$, $\Delta x q_1 q_3$ are both vacuum, $|P(x; q_1, q_2)| \leq k_3$ and $|P(x; q_1, q_3)| \leq k_2$. Then the conclusion of Theorem 1 holds.*

Proof. For two distinct points a, b in the plane, let $\overline{ab}$ denote the open line segment connecting a and b. We fix a point $y \in \mathbb{R}^2$ so that $\overline{xy}$ contains q_1. Take $p_2' \in P$ as follows: if $|P(q_1; q_2, y)| \geq k_3$, set $p_2' = q_2$; otherwise take $p_2' \in P \cap H^+(q_2; q_1, x)$ so that $|P(q_1; p_2', y)| = k_3$ (note that there exists such p_2' because $|P \cap H^+(q_2; q_1, x)| \geq \mathrm{dep}(P, x) \geq k_3 + 1$). Similarly, set $p_3' = q_3$ if $|P(q_1; q_3, y)| \geq k_2$, and otherwise, take $p_3' \in P \cap H^+(q_3; q_1, x)$ so that $|P(q_1; p_3', y)| = k_2$. Also let

$$H_i := \begin{cases} H^-(q_1; x, q_i) & \text{if } p_i' = q_i \\ H^+(x; q_1, p_i') & \text{otherwise} \end{cases}$$

for $i = 2, 3$. Then

$$P \cap H_2 \cap H_3 \text{ contains at lest } k_1 \text{ points of } P. \tag{4}$$

Set $p_1 = q_1$, and take p_2, p_3 so that

$$p_i \in H^+(p_i'; x, q_1) \cap ((P \cap H_2 \cap H_3) \cup \{p_2', p_3'\})$$

for $i = 2, 3$, and

$$H^+(p_; p_2, p_3) \cap \mathrm{conv}((P \cap H_2 \cap H_3) \cup \{p_2', p_3'\}) = \varnothing$$

(i.e., $\overline{p_1 p_3}$ is an edge of $\mathrm{conv}((P \cap H_1 \cap H_3) \cup \{p_2', p_3'\})$ facing p_1; see Fig. 2). Then since $|P(q_1; p_2', y)| \geq k_3$ and $|P(q_1; p_3', y)| \geq k_2$, (2) and (3) or Theorem 3 clearly hold. Moreover, since $(P \cap H_2 \cap H_3) \cup \{p_2', p_3'\}$ is contained in $H^-(p_1; p_2, p_3) \cup \{p_2, p_3\}$, (1) follows from (4). □

Next we prove another special case, which is the most essential part:

Claim 2. *The conclusion of Theorem 3 holds when $k_1 = \left\lceil \frac{n-5}{3} \right\rceil$, $k_2 = \left\lceil \frac{n-4}{3} \right\rceil$, $k_3 = \left\lceil \frac{n-3}{3} \right\rceil$.*

Proof. Let $\mathscr{D}$ be the family of all cellular polygons D containing x, and let $\mathscr{D}'$ be the family of $D' \in \mathscr{D}$ such that for any two adjacent edges $\overline{r_1 r_2}, \overline{r_2 r_3}$ of $D' (r_1, r_2, r_3 \in P)$, $|P(x; r_1, r_2) \cup P(x; r_2, r_3)| \geq k_1$ holds. Note that $\mathscr{D}'$ is non-empty set since any cellular triangle containing x belongs to $\mathscr{D}'$ by the assumption that

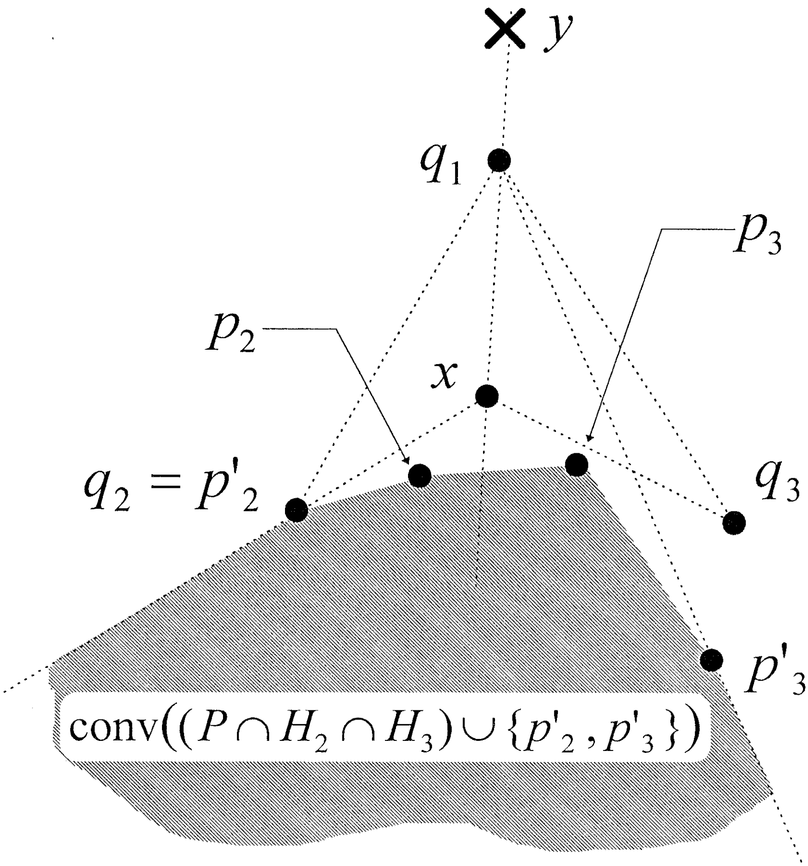

Fig. 2

$\mathrm{dep}(P, x) \geq k_3 + 1$. An edge $\overline{pq}$ of $D \in \mathscr{D}$ is said to be a *crowded edge* if $|P(x; p, q)| > k_1$, and a *deserted edge* if $|P(x; p, q)| \leq k_1$. Since $n = |P| \leq 3k_1 + 5$,

$$\text{for each } D \in \mathscr{D} \text{ the number of crowded edges of } D \text{ is at most two.} \quad (5)$$

Furthermore, if $D' \in \mathscr{D}'$, then D' has at most five edges. We henceforth assume that

$$\text{no three points of } P \text{ satisfy the assumption of Claim 2;} \quad (6)$$

in particular,

$$\text{each } D \in \mathscr{D} \text{ has no two adjacent deserted edges } \overline{r_1 r_2}, \overline{r_2 r_3} \text{ such that } \Delta r_1 r_2 r_3 \text{ contains } x. \quad (7)$$

For an edge $\overline{pq}$ of $D \in \mathscr{D}$, we say $\overline{pq}$ is a *good edge* (resp. *very good edge*) if $|P \cap H^-(x; p, q)| \geq k_1$ (resp. $\geq k_3$). It suffices to show that there exists a cellular triangle containing x two of whose edges are very good and the other one is good.

Note that since $k_3 \le k_1 + 1$, an edge of $D \in \mathscr{D}$ is very good if it is a crowded edge. Let $E_d(D)$ be the set of all deserted edges of $D \in \mathscr{D}$. We henceforth fix $D_0 \in \mathscr{D}'$ so that $f(D_0) := \sum_{\overline{pq} \in E_d(D_0)} \angle pxq$ is maximum.

Case 1. D_0 is a triangle (see Fig. 3a).

Let $r_1, r_2, r_3 \in P$ be the vertices of D_0. In view of (5) and (7), D_0 has two crowded edges, which are at the same time very good edges, and a deserted edge. By symmetry, we may assume that $\overline{r_1r_2}$ and $\overline{r_2r_3}$ are crowded edges, and $\overline{r_3r_1}$ is a deserted edge. We have only to show that $\overline{r_3r_1}$ is a good edge. We first prove $P(r_1; r_2, r_3) = \emptyset$. Suppose that $P(r_1; r_2, r_3) \ne \emptyset$. Then we can take $s \in P(r_1; r_2, r_3)$ so that the quadragon $D_1 = r_1r_2sr_3$ is cellular. Applying (5) to D_1, we see that $\overline{r_2s}$ or $\overline{sr_3}$ must be a deserted edge. If $D_1 \in \mathscr{D}'$, this contradicts the maximality of $f(D_0)$. Thus $D_1 \notin \mathscr{D}'$, which means that $\overline{sr_3}$ is a deserted edge and $|P(x; s, r_3) \cup P(x; r_3, r_1)| \le k_1 - 1$ holds. Applying (7) to D_1, we also see that Δr_1r_3s does not contain x, and hence Δr_1r_2s is a member of $\mathscr{D}'$ with a deserted edge $\overline{sr_1}$, which again contradicts the maximality of $f(D_0)$. Consequently, $P(r_1; r_2, r_3) = \emptyset$. Let l be the line passing through x and parallel to $\overline{r_1r_2}$ and H be the closed half-plane which is bounded by l and contains r_3. Then $(P \cap H) - \{r_3\} \subseteq P \cap H^-(x; r_3, r_1)$ and hence

$$\begin{aligned} |P \cap H^-(x; r_3, r_1)| &\ge |P \cap H| - 1 \\ &\ge \operatorname{dep}(P, x) - 1 \ge k_3, \end{aligned}$$

i.e., $\overline{r_3r_1}$ is a (very) good edge.

Case 2. D_0 is a quadrangle

By (7), D_0 has at least one crowded edge. In view of (5), there are two subcases to be considered.

Subcase 2.1. D_0 has one crowded edge and three deserted edges (see Fig. 3b).

Let $\overline{r_1r_2}$ be the crowded edge and let $\overline{r_2r_3}$, $\overline{r_3r_4}$, $\overline{r_4r_1}$ be the deserted edges. Again by (7), $\Delta r_1r_3r_4$ and $\Delta r_2r_3r_4$ does not contain x, which immediately implies that $\Delta r_1r_2r_3$ and $\Delta r_1r_2r_4$ contain x. Since $\overline{r_1r_2}$, $\overline{r_1r_3}$, $\overline{r_2r_4}$ are very good edges (of $\Delta r_1r_2r_3$ or $\Delta r_1r_2r_4$), it suffices to show that $\overline{r_2r_3}$ or $\overline{r_4r_1}$ is a good edge. If $P(r_3; r_2, r_4) \cap P(r_4; r_1, r_3) \ne \emptyset$, then there exists $s \in P(r_3; r_2, r_4) \cap P(r_4; r_1, r_3)$ such that $sr_2r_3r_4r_1$ is a cellular pentagon and, arguing as in Case 1, we see that one of $sr_2r_3r_4r_1$, $sr_2r_3r_4$ or $sr_3r_4r_1$ yields a contradiction to the maximality of $f(D_0)$. Thus $P(r_3; r_2, r_4) \cap P(r_4; r_1, r_3) = \emptyset$. Therefore, $P \subset P(x; r_3, r_4) \cup H^-(x; r_2, r_3) \cup H^-(x; r_4, r_1) \cup \{r_1, r_2, r_3, r_4\}$. Since $\overline{r_3r_4}$ is a deserted edge, $|P \cap H^-(x; r_2, r_3)| \ge k_1$ or $|P \cap H^-(x; r_4, r_1)| \ge k_1$ holds, which settles this subcase.

Subcase 2.2. D_0 has two crowded edges and two deserted edges (see Fig. 3c).

By the definition of $\mathscr{D}'$, the two deserted edges cannot be adjacent. Let $\overline{r_1r_2}$ and $\overline{r_3r_4}$ be the deserted edges, and let $\overline{r_2r_3}$ and $\overline{r_4r_1}$ be the crowded edges. Without loss of generality, we may assume $\Delta r_1r_2r_3$ contains x. If $|P \cap H^-(x; r_1, r_2)| \ge k_3$, $\Delta r_1r_2r_3$ clearly has three very good edges. Thus we may assume $|P \cap H^-(x; r_1, r_2)| \le k_3 - 1$. On the other hand, again as in Case 1, we see that $H^+(x; r_1, r_2) \cap$